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Hadamard Design and Artificial Neural Nets

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Hadamard theory is shown to play an important role in the generation of Boolean decision functions, a fundamental tool in the field of artificial neural network design. Based on a group-theoretic introduction of a complete set of Hadamard vectors, whose matrices are of the order of a power of two, we classify subsets according to the degree of their linear dependence. We show in the thermodynamic limit that essentially the whole Hadamard space is occupied by representatives with defect not exceeding two or three.

KEY WORDS: Neural networks; Boolean functions; Hadamard vectors; combinatorial design.

1. INTRODUCTION

Hadamard matrices have long been known in the theory of combinatorial mathematics as well as in numerous applications in technology.[1] They are widely used in design and coding theory, particularly in the context of the transmission of huge amounts of data through a noisy medium, where they serve as powerful discrete fast-Fourier transforms.[2] One of their most prominent applications was realized in 1971, when they served as codewords for the Mars Mariner 69 telemetry system in order to send photographs of Mars back to Earth.[3] Specific applications in weighing designs, where the measure of several objects is the sum or a linear combination of the individual weights, reveal that mean square weighing errors can be substantially reduced. Hadamard transforms have also been of considerable importance in problems of visual pattern recognition as well as for the construction of suitable error masks in spectroscopy.[4] Furthermore, there is promising application potential for the analysis of

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genetic algorithms operating on different coding-function combinations.[5] Most recently there has been growing interest in the theory of information processing in artificial neural networks,[6–8] where Hadamard vectors play a fundamental role in generating Boolean functions. In problems of information storage their powerful internal symmetries can be exploited in order to arrive at optimal network structures.[9]

2. A GENERAL MODEL OF DISCRETE AUTOMATA

We consider an arbitrary system of N interconnected automata only capable of taking the value $+1$ or -1, respectively. We assume that each unit can be influenced by a subset of K other units of the network. The dynamical rule governing the deterministic time evolution of the system, that is, whether unit i will be on or off at the next time step, is

$$\sigma_i(t+1) = f_i(\sigma_{j_1(i)}(t), \sigma_{j_2(i)}(t), \ldots, \sigma_{j_K(i)}(t)), \qquad i = 1, \ldots, N \tag{2.1}$$

Thus, the state of unit i depends only on the value of its input variables $\sigma_{j_1(i)}(t), \sigma_{j_2(i)}(t), \ldots, \sigma_{j_K(i)}(t)$ at the previous time step. To be general, the function f_i is represented by one of the 2^{2^K} possible Boolean rules specified by a 2^K-element truth table, a 2^K-digit binary number, or the corresponding decimal number between 0 and $2^{2^K}-1$.

Biologically motivated models may define the transition functions f_i by a set of thresholds and synaptic weights describing the strength of excitatory and inhibitory so-called higher-order interactions such that the dynamical rule (2.1) takes the special algebraic form

$$\sigma_i(t+1) = \operatorname{sgn}\left[c_i + \sum_{j_1} c_{ij_1}\sigma_{j_1}(t) + \sum_{j_1<j_2} c_{ij_1j_2}\sigma_{j_1}(t)\,\sigma_{j_2}(t) + \cdots + \sum_{j_1<\cdots<j_K} c_{ij_1\cdots j_K}\sigma_{j_1}(t)\cdots\sigma_{j_K}(t) \right] \tag{2.2}$$

The bracket term in (2.2), described by a polynomial of degree K in the K input variables, can be interpreted as the internal stimulus felt by unit i. The zeroth-order interaction described by the quantity c_i represents a threshold, while the tensor element $c_{ij_1\cdots j_s}$, usually not symmetric with respect to interchange of the subscripts, defines the weight of an sth-order interaction between model unit i and the s-subset $j_1, \ldots, j_s$. Restricted to *low* connectivity ($K \leqslant 3$), Kauffman[10] applied the model with multicell interactions and randomly chosen Boolean functions f_i to model the complex genetic regulatory system that guides cell differentiation in embryonic development. Completely or partially interconnected models restricted to two-cell interactions are often used to mimic storage and retrieval

dynamics of associative memories. One objective of this study is to show that Hadamard theory provides the decisive link between the logic formulation, Eq. (2.1), and the algebraic formulation, Eq. (2.2). We also remark that quite general models in the spirit of both formulations have been elegantly studied within group-theoretic considerations by Caianiello.[11]

3. GROUP-THEORETIC INTRODUCTION OF HADAMARD MATRICES

In order to enumerate all possible input combinations for a binary K-input model, let us first define the binary representation of the integers $0, 1, 2, \ldots, 2^K - 1$:

$$i = \sum_{l=0}^{K-1} \varepsilon_i^l 2^{K-1-l} \qquad (i = 0, 1, 2, \ldots, 2^K - 1) \tag{3.1}$$

where the most significant bit is the first one. The coefficients ε_i^l then define the binary matrix

$$E_K = (\varepsilon_i^j) \in \mathcal{M}_{2^K, K}(\{0, 1\}) \tag{3.2}$$

which, for example, for $K = 3$ reads

$$E_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix} \tag{3.3}$$

The rows ε_i of Eq. (3.3) represent a complete enumeration of *all* possible 2^K binary input combinations for a unit and could also be interpreted as all possible spin orientations of a K-spin model. The set $V_K = \{\varepsilon_0, \varepsilon_1, \ldots, \varepsilon_{2^K-1}\}$ of all row vectors consists of all 2^K distinct K-tuples over $\{0, 1\}$. Under the operation of componentwise addition modulo two, denoted as $\oplus$ (XOR), the set V_K obviously forms an elementary Abelian group of order 2^K, or, what amounts to the same thing, a K-dimensional vector space over the field $\mathbb{F}_2$ of two elements. We introduce the "decimal" representative $\mathrm{dec}(i \oplus j)$ via

$$\varepsilon_i \oplus \varepsilon_j = \varepsilon_{\mathrm{dec}(i \oplus j)} \tag{3.4}$$

The vectors ε_{2^i} $(i=0, 1,..., K-1)$ are the K canonical basis vectors e_i. Since the matrix E is of maximal rank K, the K column vectors ε^j of E_K can serve as a basis in order to generate the K-dimensional Hadamard space $\mathscr{H}_K$, a subspace of order 2^K within the space of all possible 2^{2^K} 2^K-dimensional vectors. The recipe is as follows: each K-dimensional row vector $\varepsilon_j \in \{0, 1\}^K$ is assigned a so-called Hadamard vector $h^j \in \{0, 1\}^{2^K}$ via

$$h^j = \Phi_{\mathrm{H}}(\varepsilon_j) = \sum_{l=0}^{K-1} \varepsilon_j^l \varepsilon^l \tag{3.5}$$

Thus, Φ_{H} defines a fundamental isomorphism between the K-dimensional vector space $\mathbb{F}_2^K$ and the K-dimensional Hadamard space $\mathscr{H}_K \subset \mathbb{F}_2^{2^K}$. From the algebraic point of view, this isomorphism is the canonical representation of the dual space $\mathrm{Hom}(\mathbb{F}_2^K, \mathbb{F}_2)$ as subset of the space of all functions

$$\mathrm{Maps}(\mathbb{F}_2^K, \mathbb{F}_2) = \mathbb{F}_2^{\mathbb{F}_2^K} = \mathbb{F}_2^{2^K}$$

In this representation h^j is the vector dual to ε_j. Note that the matrix elements ε_i^j of E_K also play the role of the coefficients. Since the row vectors ε_{2^l} are the canonical basis vector e_l $(l=0,..., K-1)$, we find

$$h^{2^l} = \varepsilon^l \qquad (l=0,..., K-1) \tag{3.6}$$

According to Eq. (3.5), the total set of Hadamard vectors is specified by all possible linear combinations of the K basis vectors $h^{2^l} = \varepsilon^l$ and the Hadamard matrix H_K, defined by the matrix elements

$$h_i^j = \sum_{l=0}^{K-1} \varepsilon_j^l \varepsilon_i^l \tag{3.7}$$

represents the *Gram* matrix specified by the scalar products of *all* 2^K possible distinct K-tuples given by the rows of E_K. The matrix H_K is symmetric and for $K=3$, e.g., H_K takes the form

$$H_3 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 \end{pmatrix} \tag{3.8}$$

Addition of Hadamard vectors h^i, h^j naturally gives [according to Eqs. (3.4) and (3.5)]

$$h^i \oplus h^j = \begin{cases} h^{\mathrm{dec}(i \oplus j)} & \text{for } i \neq j \\ 0 & \text{for } u = j \end{cases} \tag{3.9}$$

4. SPIN NOTATION

The isomophism between the additive cyclic group $\{0, 1, \oplus\}$ and the multiplicative group $\{1, -1, \odot\}$ leads to the standard Hadamard description commonly found in the literature.[1] In analogy to Eq. (3.5), a Hadamard vector h^j is generated by the product form

$$h^j = \prod_{l=0}^{K-1} (\varepsilon_{\odot}^l)^{\varepsilon_j^l} = \prod_{l=0}^{K-1} (h_{\odot}^{2^l})^{\varepsilon_j^l} \tag{4.1}$$

with the $\{1, -1, \odot\}$ representation $\varepsilon_{\odot}^l$ of ε^l given by

$$(\varepsilon_{\odot}^l)_i = (-1)^{\varepsilon_i^l} \tag{4.2}$$

The vector factorization, also called the Hadamard product, is defined as componentwise vector multiplication. The internal symmetries of Hadamard matrices are visualized in Fig. 1, where full and open squares correspond to 0 and 1 in the additive and to $+1$ and -1 in the multiplicative notation, respectively.

According to the construction, apart from the first row h_0, each row h_i of H_K contains 2^{K-1} entries with value -1 as well as 2^{K-1} entries with value $+1$. Furthermore, any two arbitrary vectors h_i and h_j satisfy the orthogonality condition

$$\sum_{l=0}^{2^K-1} h_i^l h_j^l = \begin{cases} 0 & \text{for } i \neq j \\ 2^K & \text{for } i = j \end{cases} \tag{4.3}$$

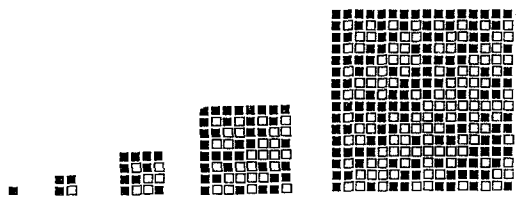

Fig. 1. Hadamard matrices for (from left to right) $K = 0, 1, 2, 3, 4$.

which implies that

$$H_K H_K = 2^K I \tag{4.4}$$

such that H_K is a matrix of Hadamard type as commonly defined in the literature.[1-4]

An equivalent design for Hadamard matrices of order 2^K with entries $h_i^j \in \{-1, +1\}$ or $h_i^j \in \{0, 1\}$ is well known from the following *recursive* construction for a maximal set of orthogonal vectors:

$$H_{l+1} = \begin{pmatrix} H_l & H_l \\ H_l & -H_l \end{pmatrix} \qquad \text{for} \quad l = 0, 1, ..., K-1 \tag{4.5}$$

with $H_0 = 1$. Note, however, that this introduction appears to be less accessible to group-theoretic studies.

5. BOOLEAN FUNCTIONS AND HADAMARD SETS

The capability of implementing arbitrary Boolean functions is of fundamental interest for neural network design based on *binary* decision elements. In this context adaptive information processing eventually aims at the determination of appropriate Boolean rules which match best a context-dependent task.

Let us now show that the algebraic description of Boolean rules according to Eq. (2.2) can be given in terms of Hadamard design in the $\{1, -1, \odot\}$ description.

Given an arbitrary 2^K-dimensional vector $c \in \mathbb{R}^{2^K}$, we consider the Hadamard coordinate transformation

$$Hc = \sum_{j=0}^{2^K-1} c_j h^j \tag{5.1}$$

Since each Hadamard vector can be expressed as a Hadamard product of the base [see Eq. (4.1)], we have

$$Hc = \sum_{j=0}^{2^K-1} c_j \prod_{l=0}^{K-1} (h^{2^l})^{\varepsilon_j^l} \tag{5.2}$$

which is nothing else but a polynomial in the basis vectors $h^{2^l} = \varepsilon^l$ with coefficients $c_j \in \mathbb{R}$. We use the following notation for this polynomial: According to Eq. (3.1), the indices $0 \leqslant j \leqslant 2^K - 1$ are identified with their binary representation ε_j. Moreover, arbitrary 0–1-vectors are represented by their support, that is, by the set of indices with entry 1. In this way, a

set $M \subset \{1,..., K\}$ corresponds to the number j_M having in its binary representation entry 1 exactly at the places given by M, that is,

$$M = \{j_1 < \cdots < j_s\} \text{ corresponds to } j_M = 2^{j_1 - 1} + 2^{j_2 - 1} + \cdots + 2^{j_s - 1} \tag{5.3}$$

Using this identification, we write c_M instead of c_j and so we get the polynomial representation

$$Hc = p_K(h^{2^0},..., h^{2^{K-1}}) = p_K(\varepsilon^0,..., \varepsilon^{K-1}) \tag{5.4}$$

with the polynomial of degree K in K spin variables $\sigma_1,..., \sigma_K$ introduced by the bracket term of Eq. (2.2):

$$p_K(\sigma_1,..., \sigma_K) = c + \sum_{j_1} c_{j_1}\sigma_{j_1} + \sum_{j_1 < j_2} c_{j_1 j_2}\sigma_{j_1}\sigma_{j_2} + \cdots + \sum_{j_1 < \cdots < j_s} c_{j_1 \cdots j_K}\sigma_{j_1}\cdots\sigma_{j_K} \tag{5.5}$$

For example, for $K = 3$, Eq. (5.5) reads

$$p_3(\sigma_1, \sigma_2, \sigma_3) = c + c_1\sigma_1 + c_2\sigma_2 + c_{12}\sigma_1\sigma_2 + c_3\sigma_3 + c_{13}\sigma_1\sigma_3 + c_{23}\sigma_2\sigma_3 + c_{123}\sigma_1\sigma_2\sigma_3 \tag{5.6}$$

Formula (5.4) finally bridges the gap between Hadamard vectors and Boolean functions defined in the algebraic notation, Eq. (2.2). Any 2^K-dimensional vector c defines a Boolean function, specified as the 2^K-dimensional output vector f according to

$$f := \operatorname{sgn}(Hc) = \operatorname{sgn}(p_K(\varepsilon^0,..., \varepsilon^{K-1})) \tag{5.7}$$

Since H the multiplicative version of the Hadamard matrix) is regular, any of the 2^{2^K} possible Boolean functions defined by f has a representation of this polynomial form.

The inverse problem, where an arbitrary Boolean function, specified by f, is to be described by Eq. (2.2) is readily solved by the matrix equation

$$Hc = 2^K \cdot Df \tag{5.8}$$

where D is a diagonal matrix with only positive entries and the factor 2^K merely serves for normalization. According to Eq. (4.4), we have

$$c = H \cdot Df \tag{5.9}$$

In this way, the problem of generating arbitrary Boolean functions is completely linearized and the components of the coefficient vector c are linear combinations of Hadamard vectors.

We note that the diagonal matrix D can be arbitrarily chosen to suit particular needs. If we choose D as the identity matrix, the coordinates of c only take the discrete values of the set $\{-2^K, -2^K+2,..., 2^K\}$. A variety of other useful problems with application potential arise. One might ask how to choose D to generate coefficient vectors c of special type, e.g., with a maximum number of zeros, corresponding to a minimal set of Hadamard subsets. Since linearly separable Boolean functions can be generated exclusively by Hadamard *basis* vectors, one could attack various counting problems within this special highly restricted and important subset. One might generate specific classes of Boolean functions characterized by ternary coefficients $c, c_1,..., c_{12\ldots K} \in \{0, +1, -1\}$. We remark that this special class is of enormous interest for economic chip design, since one needs only one single bit per synaptic connection. Obviously many of these problems are intimately related to the important problem of minimal representations for Boolean functions.

6. LINEAR RELATIONS WITHIN HADAMARD SETS

Hadamard subsets have been the focus of several recent studies within the neural network field. Under the severe restriction to first-order networks, Eq. (2.2), Krisement[6] first presented analytical results about the quality of storage of Hadamard vectors, so-called Hadamard patterns. At a later stage, Folk and Kartashov[7] gave a heuristic study on equivalent first-order network models embedding specific Hadamard sets. This work was commented upon and supplemented by Brawley and Lisonek,[8] who gave a group-theoretic approach to the classification problem without any link to neural network design. Furthermore, we want to point out that from the information storage point of view it might appear rather awkward to restrict oneself to first-order nets in problems when information is highly correlated and structured as in the case for Hadamard patterns. In fact, first-order networks only capture two-component correlations within a pattern set. However, precise first-order studies[7,8] can serve as a useful tool in order to shed some light on the poorly understood relationship between random and orthogonal pattern sets in the large-N limit. On the other hand, second-order networks, capturing three-component correlations, open the way to integrate the powerful internal symmetries of Hadamard patterns into network architectures in order to optimize economic network design as well as performance.[9]

Let us now consider a p-subset of the full Hadamard space,

$$H_p = \{h^{j_1}, h^{j_2},..., h^{j_p}\} \subset H^* = H \backslash \{0\} \tag{6.1}$$

Although the individual members of H_p are pairwise *orthogonal* with respect to $\{1, -1, \odot\}$, they are in general *not* linearly independent *within the vector space* $\mathscr{H}_K$. Thus, all

$$\binom{2^K - 1}{p}$$

subsets $H_p = \{h^{j_1}, h^{j_2}, \ldots, h^{j_p}\}$ can be classified with respect to the degree of their linear dependence: We call two sets H_p and H_p' equivalent if and only if they span vector spaces of the same dimension r,

$$\operatorname{rank}(h^{j_1}, h^{j_2}, \ldots, h^{j_p}) = \operatorname{rank}(h^{j_1'}, h^{j_2'}, \ldots, h^{j_p'}) = r \tag{6.2}$$

i.e., H_p and H_p' are of the same rank r. Because of the duality [see Eq. (3.5)] between h^j and ε_j, Eq. (6.2) is equivalent to

$$\operatorname{rank}(\varepsilon_{j_1}, \varepsilon_{j_2}, \ldots, \varepsilon_{j_p}) = \operatorname{rank}(\varepsilon_{j_1'}, \varepsilon_{j_2'}, \ldots, \varepsilon_{j_p'}) = r \tag{6.3}$$

Thus, the "larger" Hadamard space is no longer relevant. According to the equivalence relation (6.2), the space of all subsets H_p splits into orbits whose length is of considerable interest for various fields in theory and neural network applications, i.e., counting of equivalent Hadamard sets[7,8] as well as generating and classification of Boolean functions.

The length of these orbits, specified as $O_K(p, r)$, is composed of two factors, namely the number of different r-dimensional subspaces within the K-dimensional space $\mathscr{H}_K$, denoted as $M_K(r)$, times the number of subsets H_p of order p generating a *fixed* subspace of dimension r, denoted as $N(p, r)$. While the first number is independent of p, the latter does not depend on K, and the length of these classes is given by

$$O_K(p, r) = N(p, r)\, M_K(r) \tag{6.4}$$

Note that by introducing the rank r, we separate the influence of K and p. The number of subspaces of dimension r in some K-dimensional space over $\mathbb{F}_2$ is easily counted,[12]

$$M_K(r) = \prod_{i=0}^{r-1} \frac{2^K - 2^i}{2^r - 2^i} \tag{6.5}$$

[In group-theoretic terms this number is the index of the stabilizer group of any r-dimensional space V within the full linear group $\mathrm{GL}_K(\mathbb{F}_2)$.] The total number $N(p, r)$ of *all* p-subsets of $\mathbb{F}_2^r \backslash \{0\}$ generating the r-dimensional vector space $\mathbb{F}_2^r$ is given by[13]

$$N(p, r) = S(p, r) \cdot \frac{1}{p!} \cdot \prod_{i=0}^{p-1} (2^p - 2^i) \tag{6.6}$$

Here $S(p, r)$ is specified by a q-analogon of the recurrence relation for the Stirling numbers of the first kind.[13]

6.1. Orbit Lengths for $r = p$

In the case $r = p$ the quantity $N(p, p)$ is the number of unordered bases of a p-dimensional vector space, hence given as the number of ordered bases $\#\mathrm{GL}_p(\mathbb{F}_2) = \prod_{i=0}^{p-1}(2^p - 2^i)$ divided by $p!$. [This is in accordance with Eq. (6.6), since $S(p, p) = 1$.] For the orbit of a linearly independent p-Hadamard set we therefore find

$$O_K(p, p) = N(p, p) \cdot M_K(p) = \frac{1}{p!} \prod_{i=0}^{p-1} (2^K - 2^i) \tag{6.7}$$

Equation (6.7) immediately reveals that for $K \to \infty$ and finite p the ratio

$$O_K(p, p) \Big/ \binom{2^K - 1}{p} = \prod_{i=0}^{p-1} \frac{2^K - 2^i}{2^K - i} \tag{6.8}$$

approaches the value 1 such that this orbit practically occupies the whole volume. We will see in the next section that for the more interesting case $p = K$ the corresponding limit will emerge in a natural way.

6.2. Orbit Length for Defect >0

In the general case we introduce for convenience the defect $d := p - r$. According to Eqs. (6.4)–(6.6) the length of the orbit consisting of all p-subsets of rank r is given by

$$O_K(p, p-d) = \frac{S(p, p-d)}{p!} \cdot \prod_{i=0}^{p-1} (2^p - 2^i) \cdot \prod_{i=0}^{p-d-1} \frac{2^K - 2^i}{p^{p-d} - 2^i} \tag{6.9}$$

In order to study the special case $K = p$, we first define the ratio $R(p, d) = O_p(p, p-d)/O_p(p, p)$, which can be written as

$$R(p, d) = \frac{S(p, p-d)}{2^{d(p-d)}} \cdot \prod_{i=0}^{d-1} \frac{1}{2^p - 2^i} \tag{6.10}$$

Note that the ratio $R(p, d)$ specifies the fractional space of the volume filled by the orbit with defect d compared to the space filled by the orbit with defect $d = 0$. Since for higher values of d the generalized Stirling

numbers[13] $S(p, p-d)$ are not given explicitly, we exploit the fact that in the large-p limit and d finite they exhibit the asymptotic behavior

$$S(p, p-d) = 2^{dp-\binom{d}{2}} \prod_{i=1}^{d} \frac{1}{(2^i-1)} \tag{6.11}$$

Thus, for $p \gg d$ the quantity $R(p, d)$ takes the simple form

$$R(p, d) = \frac{2^d}{\prod_{i=1}^{d} (2^i-1)^2} \tag{6.12}$$

where $R(p, 0) = 1$. The rapidly converging series

$$\sum_{l=0}^{\infty} R(p, l) = \sum_{l=0}^{d} R(p, l) + \sum_{l=d+1}^{\infty} R(p, l) \tag{6.13}$$

can be easily calculated within arbitrary accuracy, since for the remainder $D(d)$ one finds the following inequality serving as a strict upper bound:

$$D(d) = \sum_{l=d+1}^{\infty} R(p, l) = \sum_{l=d+1}^{\infty} \frac{2^l}{\prod_{i=1}^{l} (2^i-1)^2} < \frac{1}{2^{(d+1)^2}-1} \tag{6.14}$$

Hence we have

$$\sum_{l=1}^{\infty} R(p, l) = 3.4627466 \tag{6.15}$$

The individual ratios $R(p, d)$ representing the fractional space of the orbit with defect d compared to that with defect 0 take the values 2, 4/9, and 8/441 for $d = 1$, 2, and 3, respectively. Since the quantities $O(p, p-d)$ satisfy the sum rule

$$\sum_{d=0}^{p-2} O(p, p-d) = \binom{2^p-1}{p} \tag{6.16}$$

we find for the corresponding fractional volumes with respect to the total volume

$$\sum_{l=0}^{d} \left[O_p(p, p-l) \Big/ \binom{2^p-1}{p} \right] = \begin{cases} 0.288788 & \text{for } d=0 \\ 0.866364 & \text{for } d=1 \\ 0.994716 & \text{for } d=2 \\ 0.999953 & \text{for } d=3 \end{cases} \tag{6.17}$$

Thus, choosing a random set of p Hadamard patterns, there is—in the limit $K = p \to \infty$—only a 0.0047% chance that their defect d is greater than

three. Accordingly, the study of finer-structured equivalence classes[7,8] that are well separated by different defects could be restricted to defects less than or equal to three. Note that the fractional volume for the linearly independent orbit in Eq. (6.17) ($d=0$) can also directly be calculated from Eq. (6.8); however, the product series converges rather slowly, in marked contrast to Eq. (6.13).

In studying the finer-structured orbits of p-subsets H_p under the action of the full linear group $GL_K(\mathbb{F}_2)$ suggested by Folk and Kartashov,[7] the defect d will also play a vital role.[16] We also indicate that, there, too, the influence of K and of p may be separated with the help of the rank r. This reduces the problem to the study of the r-dimensional general linear group $GL_r(\mathbb{F}_2)$, and then by introducing the defect d one further reduces the investigation to the d-dimensional linear group $GL_d(\mathbb{F}_2)$.

7. CONCLUSION AND OUTLOOK

We have demonstrated that Hadamard subsets can serve as natural generators for specific designs of arbitrary Boolean functions. Moreover, simple counting methods allow the classification of Hadamard subsets according to the rank of their generators. Further investigations involving the study of finite group representations could shed some new light on some old and open problems of information storage based on Boolean input–output rules.[15] Moreover, fine-structured classifications according to refs. 7 and 8 could give some substantial contributions to the intriguing fundamental problem of the economic generation and classification of Boolean functions.

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